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LETTER TO THE EDITOR

On the topology of the domain of outer communication

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Abstract. It is shown that the topological censorship theorem of Friedman, Schleich and Witt implies, in the general setting of their result, that the domain of outer communication $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ is simply connected. This improves recent related results of Chruściel and Wald and of Jacobson and Venkataramani.

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Chruściel and Wald [CW] and Jacobson and Venkataramani [JV] have used the topological censorship theorem of Friedman, Schleich and Witt [FSW] to obtain restrictions on the topology (expressed in terms of standard topological invariants) of the domain of outer communication $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$. Jacobson and Venkataramani show, in the general setting of [FSW], that the first homology group of a Cauchy surface for the domain of outer communication (actually the portion of it to the future of a cut K on $\mathcal{I}^-$) must be finite. In the special case of stationary black hole spacetimes, Chruściel and Wald obtain a stronger conclusion, namely that the domain of outer communication (which is assumed to be globally hyperbolic) is simply connected. Jacobson and Venkataramani, however, describe certain physical mechanisms which, they suggest, might prevent an approach like that taken in [CW] from working in general.

In this note we present a simple proof, by a different method, that, in the general setting of the FSW topological censorship theorem, the domain of outer communication must indeed be simply connected.

We first recall the statement of the FSW topological censorship theorem. Let M be a spacetime with Lorentzian metric g. In [FSW] M is assumed to be asymptotically flat in the sense of admitting a suitably well behaved past and future null infinity, $\mathcal{I}^-$ and $\mathcal{I}^+$, respectively. Hence, in particular, there exists a spacetime-with-boundary M_0, with metric g_0, and a smooth function $\Omega : M_0 \to \mathbb{R}$ such that $M \subset M_0$, $g_0 = \Omega^2 g$ on M, $\Omega = 0$ and $d\Omega \neq 0$ (pointwise) on ∂M_0, $\partial M_0 = M_0 \setminus M = \mathcal{I} = \mathcal{I}^- \sqcup \mathcal{I}^+$, where $\mathcal{I}^-$ and $\mathcal{I}^+$ (past and future null infinity, respectively) are null hypersurfaces with topology $S^2 \times \mathbb{R}$.

Let U be a connected and simply connected neighborhood of $\mathcal{I}$. We assume $U \cap M \subset I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$. Let γ_0 be any curve in U with initial endpoint on $\mathcal{I}^-$ and final endpoint on $\mathcal{I}^+$. In [FSW] it is assumed that γ_0 is timelike, but this is not used in an essential way, and it will be convenient not to make this assumption here.

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Theorem 1 (FSW topological censorship). Let M be a globally hyperbolic asymptotically flat spacetime which satisfies the null convergence condition $\mathrm{Ric}(K, K) \geq 0$ for all null vectors K. Then every causal curve γ from $\mathcal{I}^-$ to $\mathcal{I}^+$ is homotopic to γ_0 (relative to $\mathcal{I}$).

We now prove, in the setting of theorem 1, that the domain of outer communication $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ is simply connected.

Theorem 2. Let M be an asymptotically flat spacetime which satisfies the null convergence condition, and assume $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ is globally hyperbolic. Then $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ is simply connected.

It is easily seen that theorem 2 implies theorem 1, and hence provides a slightly different way of expressing topological censorship.

Proof. Without loss of generality we may assume $M = I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$. (Otherwise set $M' = I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ and work with M'.) Note that M is simply connected if and only if M_0 is simply connected, where M_0 is the 'conformal completion' of M described above.

We will prove that M_0 is simply connected. To this end, pass to the universal covering spacetime $(\tilde{M}_0, \pi)$, where $\pi : \tilde{M}_0 \to M_0$ is the covering map. With $\tilde{M}_0$ equipped with the pullback metric $\pi^* g_0$, π is a local isometry and hence preserves the causal character of curves.

Since $U \subset M_0$ is simply connected, $\tilde{U} = \pi^{-1}(U)$ consists of a disjoint union of copies of U, $\tilde{U} = \sqcup_{\alpha \in A} U_\alpha$, where $|A| =$ the number of sheets of the covering. Associated to each U_α is a null infinity $\mathcal{I}_\alpha = \mathcal{I}_\alpha^- \sqcup \mathcal{I}_\alpha^+$, where $\mathcal{I}_\alpha$ is a copy of $\mathcal{I} = \mathcal{I}^- \sqcup \mathcal{I}^+ \subset M_0$. □

Theorem 2 follows from two claims.

Claim 1. For all $\alpha \neq \beta$, $I^-(\mathcal{I}_\alpha^+) \cap \mathcal{I}_\beta^- = \emptyset$.

As the discussion in [FSW] shows (cf also [CW]) this claim is, in some sense, the key fact underlying theorem 1. It is also a consequence of theorem 1, as we now observe.

Proof of claim 1. Suppose, to the contrary, that there exists a timelike curve $\tilde{\gamma}$ from $\tilde{p} \in \mathcal{I}_\beta^-$ to $\tilde{q} \in \mathcal{I}_\alpha^+$, with $\alpha \neq \beta$. Let γ_0 be a curve in U from $p = \pi(\tilde{p}) \in \mathcal{I}^-$ to $q = \pi(\tilde{q}) \in \mathcal{I}^+$. Then, by theorem 1, $\gamma = \pi \circ \tilde{\gamma}$ is a timelike curve from p to q in M_0 homotopic to γ_0 relative to $\mathcal{I}$. Since $\mathcal{I}^-$ and $\mathcal{I}^+$ are simply connected, it follows that γ and γ_0 are fixed end point homotopic. Hence, the lift $\tilde{\gamma}_0$ of γ_0 starting at $\tilde{p}$ has the same final end point as $\tilde{\gamma}$. But this contradicts the fact that $\tilde{\gamma}_0 \subset U_\beta$ and $\tilde{q} \in \mathcal{I}_\alpha^+$.

Now fix a component U_0 of $\tilde{U} = \pi^{-1}(U)$, and let $\mathcal{I}_0 = \mathcal{I}_0^- \sqcup \mathcal{I}_0^+$ be the associated null infinity. □

Claim 2. $I^-(\mathcal{I}_0^+) \supset \bigsqcup_{\alpha \in A} \mathcal{I}_\alpha^-$.

Note that claims 1 and 2 are consistent only if $\tilde{M}_0$ is a *one-sheeted* covering of M_0 and hence only if M_0 is simply connected. Thus, to prove theorem 2, it remains to prove claim 2.

Proof of claim 2. Let $\tilde{M} = \pi^{-1}(M) = \tilde{M}_0 \setminus \partial\tilde{M}_0$. Since $\mathcal{I}^- \subset I^-(M)$, it follows that $\bigsqcup_{\alpha\in A} \mathcal{I}^-_\alpha \subset I^-(\tilde{M})$. Hence, it suffices to show that $I^-(\mathcal{I}^+_0) \supset \tilde{M}$. If this inclusion does not hold then $\partial I^-(\mathcal{I}^+_0) \cap \tilde{M} \neq \emptyset$ (since $\tilde{M}$ is connected). Pick a point $\tilde{q} \in \partial I^-(\mathcal{I}^+_0) \cap \tilde{M}$ and consider the point $q = \pi(\tilde{q}) \in M$. Since $M = I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$, there exists a past directed timelike curve γ_1 from q to $p \in \mathcal{I}^-$ and a future directed timelike curve γ_2 from q to $r \in \mathcal{I}^+$. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be the lifts from $\tilde{q}$ of γ_1 and γ_2, respectively.

Since, except for its initial point $\tilde{q}$, $\tilde{\gamma}_1$ is contained in $I^-(\mathcal{I}^+_0)$, the past end point $\tilde{p}$ of $\tilde{\gamma}_1$ must, by claim 1, be on $\mathcal{I}^-_0$. On the other hand, the future end point $\tilde{r}$ of $\tilde{\gamma}_2$ must be on a future null infinity $\mathcal{I}^+_\alpha$ distinct from $\mathcal{I}^+_0$ (since, otherwise, we would have $\tilde{q} \in I^-(\mathcal{I}^+_0)$). Thus, the timelike curve $-\tilde{\gamma}_1 + \tilde{\gamma}_2$ extends from $\mathcal{I}^-_0$ to $\mathcal{I}^+_\alpha \neq \mathcal{I}^+_0$, in contradiction to claim 1. This completes the proof of claim 2 and, in turn, theorem 2. □

Remark. The results of Chruściel and Wald [CW] and Jacobson and Venkataramani [JV] discussed at the beginning are used to obtain improvements of the classical theorem of Hawking that black hole boundaries in stationary black hole spacetimes are spherical (see also [BG] for a related result). Jacobson and Venkataramani show, under fairly general circumstances (e.g. without assuming stationarity) that sufficiently nice cross sections of the event horizon are either spheres or, without additional orientation assumptions, projective planes. Theorem 2 above, which is easily adapted to the slightly different situation considered in [JV], can be used to eliminate the possibility of the occurence of projective planes.

We make another remark regarding energy conditions. As discussed in [FSW], the energy condition in theorem 1 (and hence theorem 2) can be weakened. Note, however, that the *averaged null energy condition*, as it is defined in [FSW], must either be supplemented by the generic condition or be required to hold on future directed null *half*-geodesics (emanating from points in U).

We conclude by mentioning a variation of the proof of theorem 2 suggested by Greg Burnett (personal communication). With the notation as in the proof of theorem 2, observe that $\tilde{M} = \pi^{-1}(M)$ is the universal cover of M. Consider the collection of open subsets of $\tilde{M}$, $\ll \mathcal{I}_\alpha \gg = I^-(\mathcal{I}^+_\alpha) \cap I^+(\mathcal{I}^-_\alpha) \cap \tilde{M}$, $\alpha \in A$. Using theorem 1, one easily verifies the following alternatives to claims 1 and 2: (1) for all $\alpha \neq \beta$, $\ll \mathcal{I}_\alpha \gg \cap \ll \mathcal{I}_\beta \gg = \emptyset$, and (2) the collection $\{\ll \mathcal{I}_\alpha \gg \,:\, \alpha \in A\}$ covers $\tilde{M}$. Thus, since $\tilde{M}$ is connected, there can be only one $\ll \mathcal{I}_\alpha \gg$, and hence, as before, M is simply connected.

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